A Continuity Equation For Time Series Water Wave Modeling Formulated Using Weighted Total Acceleration

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Abstract*— *Continuity equation for wave modeling is still being developed. There are quite a lot of versions of this equation. This research formulates continuity equation in a simple form to simplify its numerical and analytical solution.
The formulation of the continuity equation is done by performing mass conservation law in a water column with free surface and by performing weighted total acceleration. Then, the continuity equation is performed along with the surface momentum equation and completed numerically to modeling one-dimensional wave dynamism. The equation is capable of modeling shoalingand breaking.
***Keywords*— Continuity Equation, Weighted total acceleration equation.**

I. INTRODUCTION

Time series water wave equation is generally called Boussinesq type equation.There are quite a lot of versions of Boussinesq equation, either its continuity equation or water surface equation as well as its momentum equation. Those equations generally consist of the second or higher differential elements which quite complicate the solution both analytically and numerically. Some researcher who have developed Boussinesq equation are among othersBoussinesq, J. (1871), Dingeman, M.W. (1997), Ham;L., Madsen, P.A., Peregrin, D.H. (1993), Johnson, R.S. (1997), Kirby, J.T. (2003), Peregrine, D.H. (1967), Peregrine, D.H. (1972) and many more.

Governing equations in this research are water surface equation and surface momentum equation, both of which use particle velocity at the surface as the variable and both are in the form of time and space differential equation in simple form. Water surface equation is formulated based on mass conservation law and by performing weighted total acceleration at the kinematic free surface boundary condition. The momentum equation is obtained by performing weighted total acceleration at the Euler momentum equation. The integration with water depth from this momentum equation produces surface momentum equation with particle velocity variable at the surface.

Both governing equations are done using numerical method where spatial differential is done using finite difference method, whereas time differential is done using corrector predictor method.

II. TOTAL DERIVATIVE EQUATION

Hutahaean (2019a) developed weighted total acceleration equation at water particle in horizontal direction, i.e.,

$$\frac{Du}{dt}=\gamma\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\ ....(1)$$

$u$is particle velocity in horizontal-$x$direction and$w$is particle velocity in vertical-z direction. Weighted total acceleration, was actually formulated for the function $f=f(x,t)$. However, in this research it is performed at $f=f(x,z,t)$, because the wave being discussed is a wave moving to horizontal-$x$direction and vertical z dimension is eliminated with the integration process, so the equation becomes a function of $f=f(x,t)$.

The changes in total water surface elevation is

$$\frac{D\eta}{dt}=\gamma\frac{\partial\eta}{\partial t}+u_\eta\frac{\partial\eta}{\partial x}\ ....(2)$$

$\eta=\eta(x,t)$is water surface elevation against still water level (Fig. 1). In (1) and (2) there is time coefficient or time scale at time differential i.e. $\gamma$with a value of 2.87-3.14 in Hutahaean (2019a,b). The value of γ is very much determined by basic equation in which the total acceleration was performed. In this research the correspond γvalue of 2.00 is found.

Based on (2), then kinematic free surface boundary condition that was formulated from total derivative equation of the changes of a surface (Dean (1991)), becomes

$w_\eta = \gamma \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$ ....(3)

$u_\eta$is the velocity of horizontal-$x$direction at the surface.

III. CONTINUITY EQUATION

3.1. The formulation of continuity equation

Continuity equation or water wave surface equation will be formulated in a water column (Fig.1.) with free water surface, where as a result of an input-output in a water column in a very small time interval $\gamma\delta t$, a change in water surface elevation of $\delta\eta$ occurs so there is also a change in the per width unit volume of $\delta\eta\delta x$. For a very small $\delta x$where$\delta x = dx$

$\delta m = \frac{\rho}{2}\delta\eta dx$ ....(4)

The change in water mass from input-output process (Fig. 1.) is,

$$\delta m = \rho\big(u - (u + \delta u)\big)\delta z\delta t + \big(w - (w + \delta w)\big)\delta x\gamma\delta t$$

$$\delta m = -\rho\left(\frac{\delta u}{\delta x} + \frac{\delta w}{\delta z}\right)\delta x\delta z\gamma\delta t$$

Total change in water mass in the water column at very small δxandδz,

$\delta m = -\rho\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz\, dx\gamma\delta t$ ....(5)

$h$is water depth against still water level, $\eta = \eta(x,t)$is the water surface elevation also against water level..For incompressible fluid, δmin (4) is the same asδmin (5),

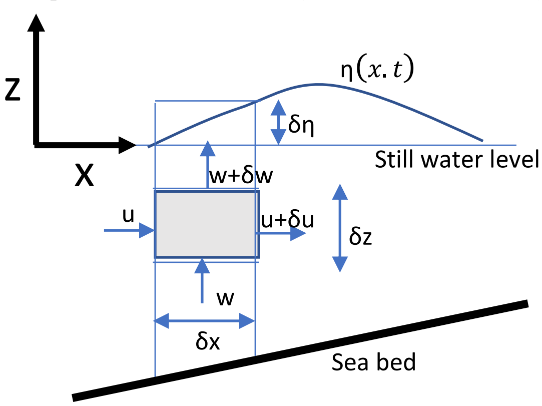

Fig. 1: Water column to formulate continuity equation.

$$\frac{\rho}{2}\delta\eta dx = -\rho\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz\, dx\gamma\delta t$$

Both sides are divided by ρ, $dx$and$\gamma\delta t$,

$$\frac{\delta\eta}{2\gamma\delta t}dx = -\rho\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz$$

For a very smallδt,

$$\frac{D\eta}{2\gamma dt} = -\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz$$

Substitute (2) to the left side of the equation

$$\frac{1}{2}\frac{\partial\eta}{\partial t} + \frac{u_\eta}{2\gamma}\frac{\partial\eta}{\partial x} = -\int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz$$

The integration of the second term of the right side is done and substituted kinematic free surface bondary condition and kinematic bottom boundary condition,

$$\left(\gamma + \frac{1}{2}\right)\frac{\partial\eta}{\partial t} = -\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz - \left(1 + \frac{1}{2\gamma}\right)u_\eta\frac{\partial\eta}{\partial x}$$

$-u_{-h}\frac{\partial h}{\partial x}$........(6)

The integration of the first term right side is performed with Leibniz integration (Protter (1985)),

$$\int_{\alpha}^{\beta}\frac{\partial f}{\partial x}dz = \frac{\partial}{\partial x}\int_{\alpha}^{\beta}f\, dz - f_\beta\frac{\partial\beta}{\partial x} + f_\alpha\frac{\partial\alpha}{\partial x}$$

Obtain,

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz = \frac{\partial}{\partial x}\int_{-h}^{\eta}u\, dz - u_\eta\frac{\partial\eta}{\partial x} - u_{-h}\frac{\partial h}{\partial x}$$

(6),becomes

$\left(\gamma + \frac{1}{2}\right)\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\int_{-h}^{\eta}udz - \frac{u_\eta}{2\gamma}\frac{\partial\eta}{\partial x}$ ....(7)

Integration in the right side in (7) is done using velocity equation from Dean (1991) and the result of integration is expressed as a function of surface horizontal velocity u_η in order to correspond with momentum equation that produces surface velocityu_η. Velocity potential equation as the result of Laplace equation solution (Dean (1991)) is

$\Phi(x,z,t) = Gcoskxcoshk(h+z)sin\sigma t$ ....(8)

Gwave constant, k wave number andσ angular frequency. Particle velocity in horizontal-x direction is

$$u(x,z,t) = -\frac{\partial\Phi}{\partial x}$$

$= Gksinkxcoshk(h+z)sin\sigma t$.......(9)

Using (9), $u = \frac{coshk(h+z)}{coshk(h+\eta)}u_\eta$, then integration in (7) becomes,

$$\frac{\partial}{\partial x}\int_{-h}^{\eta}udz = \frac{\partial}{\partial x}\int_{-h}^{\eta}\frac{coshk(h+z)}{coshk(h+\eta)}u_\eta dz$$

Completing the integration will obtain

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$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u dz = \frac{\partial}{\partial x}\left(\frac{u_\eta tanhk(h+\eta)}{k}\right)$$

From the wave-number conservation equation (Hutahaean (2019a)), $tanhk(h+\eta) = tanhk_0(h_0+\eta_0) = 1$, where k_0 is wave number in deep water, h_0 is deep water depth and η_0 is water surface elevation in deep water, can have a value of $\frac{A_0}{2}$ or others, A_0 is wave amplitude in deep water. Therefore, the result of the integration becomes,

$$\frac{\partial}{\partial x}\int_{-h}^{\eta} u dz = \frac{\partial}{\partial x}\left(\frac{u_\eta}{k}\right)$$

Substitute the result of integration to (7),

$$\left(\gamma+\frac{1}{2}\right)\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u_\eta}{k}\right) - \frac{u_\eta}{2\gamma}\frac{\partial\eta}{\partial x} \quad ......(10)$$

Equation (10) is a continuity equation that will be used in this researchor water wave surface equation in the form of differential equation. In (10), there is wave number $k$parameter that should be known, and some other characteristics that should also be known, among other is deep water depthd_0, i.e. maximumwater depth if the equation was done in water depth $d$which is bigger than d_0, so the calculation is done using d_0.Next is wave amplitude maximumA_{max}, i.e. maximum amplitude in a wave period that can be inputted to the model.

3.2. The calculation ofA_{max}andd_0.

It's been known that there is a relation between water depth $d$and wave number k, then the calculation will be easier if in (10) wave number $k$is substituted with water depth d. Whereas the equation for wave number in deep water $k_0$can be calculated using the following equation, the formulation of an equation outside the scope of this research, will be written in the next paper.

$$\gamma\left(\gamma+\frac{1}{2}\right)\sigma^2 = gk_0(1-k_0A_0) \quad ....(11)$$

$A_0$is wave amplitude which is an input, $k_0$is deep water wave number, σ is angular frequency, $\sigma = \frac{2\pi}{T}$, $T$is wave period. $k_0$in(11) can be calculated using simple calculation, i.e. finding the root of the quadratic equation. (11) can be completed if determinant $D = g^2 - 4gA\gamma\left(\gamma+\frac{1}{2}\right)\sigma^2$is bigger than or the same as zero.In case of $D = 0$,obtains

$$A_{0,max} = \frac{g}{4\gamma\left(\gamma+\frac{1}{2}\right)\sigma^2} \quad ....(12)$$

In deep water $tanhk_0\left(d_0+\frac{A_0}{2}\right) = 1$ applies. Assuming that wave amplitude is much smaller than deep water depth, or $\frac{A_0}{2d_0} \ll 1$, A_0 deep water wave amplitude andd_0 deep water depth, then the following relation applies

$$tanhk_0\left(d_0+\frac{A_0}{2}\right) = tanhk_0 d_0\left(1+\frac{A_0}{2d_0}\right)$$
$$= tanhk_0 d_0 = 1$$

$k_0$is wave number in deep water depth d_0. As deep water the following criteria is used

$$k_0 d_0 = 1.7\pi \quad ....(13)$$

where$tanh 1.7\pi = 0.999954 \approx 1$, the uses of this $1.7\,\pi$value is also based on the review of the produced breaker depth. $k_0$was obtained from (11), therefore$d_0$can be calculated using (13).

Bases on wave number conservation equation (Hutahaean (2019a)), the relation between wave number $k_d$in a depth d, with wave number in deep water is,

$k_d(h+\eta) = k_0(d_0+\eta_0)$. Assuming that $\frac{\eta_0}{d_0} \ll 1$, then

$k_d(h+\eta) = k_0 d_0 = 1.7\pi$ .Or,

$$k_d = \frac{1.7\pi}{(h+\eta)} \quad ....(14)$$

3.3. The Final Water Wave Surface Equation

By substituting (14) to (10), the final water wave equation was obtained with water depth $d$as its parameter, i.e

$$\left(\gamma+\frac{1}{2}\right)\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u_\eta(d+\eta)}{1.7\pi}\right) - \frac{u_\eta}{2\gamma}\frac{\partial\eta}{\partial x} \quad ....(15)$$

Therefore. There is no need to calculate wave number k.

An example of the calculation of deep water wavelength $L_0 = \frac{2\pi}{k_0}$, deep water depth $d_0$and$A_{0,max}$where $\gamma = 2.000$is used is presented in Table (1) below.

Table.1: The value of $A_{0,max}$, $d_0$andL_0

T (sec.)	$A_{0,max}$ (m)	d_0 (m)	L_0 (m)	$\frac{d_0}{L_0}$
6	0,447	4,929	5,798	0,85
7	0,608	6,708	7,892	0,85
8	0,794	8,762	10,308	0,85
9	1,005	11,09	13,047	0,85
10	1,241	13,691	16,107	0,85
11	1,502	16,566	19,489	0,85
12	1,787	19,715	23,194	0,85
13	2,098	23,137	27,221	0,85
14	2,433	26,834	31,569	0,85
15	2,793	30,804	36,24	0,85

IV. MOMENTUM EQUATION

Weighted total acceleration equation is done in Euler momentum equation in horizontal-$x$direction and vertical-$z$direction consecutively (Anderson (1995)),

$$\gamma\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ....(17)$$

$$\gamma\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \quad ....(18)$$

(18) is written as an equation for $p$andthe nature of irrotional flow was performed i.e.$\frac{\partial w}{\partial x}=\frac{\partial u}{\partial z}$, the equation is integrated to vertical-z axis, surface dynamic boundarycondition is performed, i.e. $p_\eta=0$, and differentiated against horizontal-x axis.

$$\frac{1}{\rho}\frac{\partial p}{\partial x}=\gamma\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz+\frac{1}{2}\frac{\partial}{\partial x}\left(u_\eta^2+w_\eta^2\right)$$
$$-\frac{1}{2}\frac{\partial}{\partial x}(u^2+w^2)+g\frac{\partial\eta}{\partial x} \quad ..... (19)$$

In (17) the nature of irrotional flow was performed, i.e. $\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$, and substitute (19) to the right side of the equation,

$$\gamma\frac{\partial u}{\partial t}=-\gamma\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz$$
$$-\frac{1}{2}\frac{\partial}{\partial x}\left(u_\eta^2+w_\eta^2\right)-g\frac{\partial\eta}{\partial x} \quad .....(20)$$

The solution of $\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz$is done using velocity potential (8), where the particle velocity in horizontal direction is in equation (9). Particle velocity in vertical-z, is

$$w=-\frac{\partial\Phi}{\partial z}=-Gksinhk(h+z)coskxsin\sigma t$$
....(21)

$$\frac{\partial w}{\partial t}=-Gksinhk(h+z)\sigma coskxcos\sigma t \quad ....(22)$$

(22) is integrated against time t

$$\int_z^\eta\frac{\partial w}{\partial t}dz=$$
$$-G\left(coshk(h+\eta)-coshk(h+z)\right)$$
$$\sigma coskxcos\sigma t$$

Differentiated against horizontal-x axis

$$\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz=$$
$$Gk\left(coshk(h+\eta)-coshk(h+z)\right)$$
$$\sigma sinkxcos\sigma t$$

Equation (9) is differentiated against time t, $\frac{\partial u}{\partial t}=Gksinkxcoshk(h+z)\sigma cos\sigma t$, which shows that this form is in $\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz$, so the following relation is obtained

$$\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz=\frac{\partial u_\eta}{\partial t}-\frac{\partial u}{\partial t}$$

Substitute this equation to (20),

$$\frac{\partial u_\eta}{\partial t}=-\left(\frac{1}{2}\frac{\partial}{\partial x}\left(u_\eta^2+w_\eta^2\right)+g\frac{\partial\eta}{\partial x}\right)\frac{1}{\gamma} \quad ....(23)$$

(23) is surface momentum equation that produces surface velocityu_η. By completing $\frac{\partial}{\partial x}\int_z^\eta\frac{\partial w}{\partial t}dz$with the method above, then in the momentum equation there is an influence of continuity equation or momentum equation that was produced and controlled by continuity equation. Another control by continuity equation on momentum equation is on variablew_η , i.e.$w_\eta=\gamma\frac{\partial\eta}{\partial t}+u_\eta\frac{\partial\eta}{\partial x}$where$\frac{\partial\eta}{\partial t}$is obtained from continuity equation. Therefore, momentum equation (20) is controlled by water surface equation.

V. RESULT OF THE MODEL

5.1. Numerical Solution

In this research, water surface equation and momentum equation are done with finite difference method for spatial differential, whereas time differential is done using predictor-corrector method based on Newton-Cote numerical integration (Abramowitz (1972)). Whereas the predictor-corrector method is as follows. As an example water surface equation (15) will be done. The water surface equation can be written in the form of,

$$\frac{\partial\eta}{\partial t}=F(t)$$

$$F(t)=-\frac{1}{\left(\gamma+\frac{1}{2}\right)}\left(\frac{\partial}{\partial x}\left(\frac{u_\eta(d+\eta)}{1.7\pi}\right)+\frac{u_\eta}{2\gamma}\frac{\partial\eta}{\partial x}\right) \quad .....(24)$$

The equation is integrated against time from $t=t-\delta t$until $t=t+\delta t$, where the integration of the right side of the equation is done with Newton-Cote numerical integration with 3 (three) integration points,

$$\int_{t-\delta t}^{t+\delta t}\partial\eta=\int_{t-\delta t}^{t+\delta t}F(t)dt$$
$$\eta^{t+\delta t}=\eta^{t-\delta t}+\delta t\left(\frac{1}{3}F^{t-\delta t}+\frac{4}{3}F^t+\frac{1}{3}F^{t+\delta t}\right)$$
....(25)

$F^{t+\delta t}$is unknown number, therefore it needs to be predicted using Taylor series and finite difference method, where the step is called predictor step, i.e.

$$F^{t+\delta t}=F^t+\delta t\left(\frac{F^t-F^{t-\delta t}}{\delta t}\right) \quad ....(26)$$

Substitute (26) to (25), the value of $\eta^{t+\delta t}$prediction can be calculated. With similar way, momentum equation is done, and $u_\eta^{t+\delta t}$prediction is obtained. With those prediction values,$F^{t+\delta t}$can be calculated with (24), and (25) is done. This step is called corrector step. This corrector step is also

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done on momentum equation interchangeably with the water surface equation, and repeated until a convergence is reached where $\left|\eta_{new}^{t+\delta t}-\eta_{old}^{t+\delta t}\right| \leq$ εand$\left|u_{\eta,new}^{t+\delta t}-u_{\eta,old}^{t+\delta t}\right| \leq$ εwhereεis a very small positive number as an iteration accuracy criteria. Generally, a convergence is reached with 5 iterations.

5.2. The Result of Model Execution

a. In constant depth of 11.0 m

In this section the model is done in a channel with a constant water depth of $d = 11.00$ m, with wave period of 8 sec., wave amplitude $A_0 = 0.794$ m, where actually that does not mean that wave height is twice that of wave amplitude, Hutahaean(2019 c).Deep water depth for this wave is $d_0 = 8.762$m. In the case that $d$is bigger than$d_0$then the calculation of$(d+\eta)$in (15),$(d_0+\eta)$ is used. The model is done using two boundary conditions, i.e. closed-end boundary condition where horizontal velocity$u = 0$, whereas in the opened-endthe model was given an input, i.e. sinusoidal wave$\eta_0 = A_0 sin\sigma t$. The input is done only for 1 time wave period.

The result of the execution for 8, 24, and 40 sec. is presented in (Fig.2.). In the execution for 8 sec., the wave profile is still in the form of sinusoidal, but the wave trough elevation is smaller than the elevation. In the execution for 24 sec, the formed wave trough is getting smaller and farther away, similarly with execution for 40 sec, the wave trough is getting smaller and farther away where water ripple is formed and the form of the main wave is a perfect cnoidal wave or more accurately it is called solitary wave.

As a conclusion of the model execution in this constant water depth is that in the deep water, the equation used produced perfect cnoidaltype wave or also can be called as solitary wavetype, even though the input of sinusoidal wave, the wave trough part disappears.

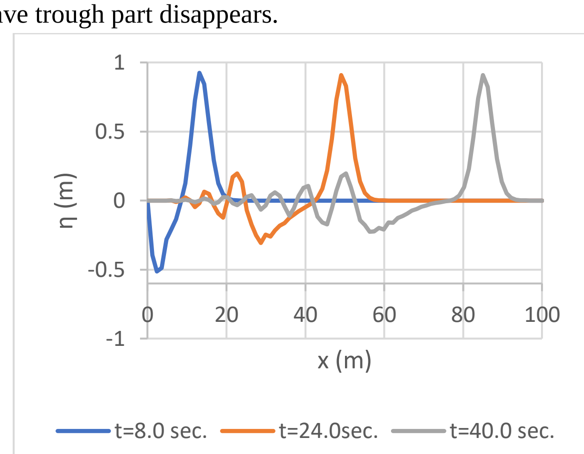

Fig.2.Wave profile after the execution for 8,24 and 40 sec.

b. In a changing depth

With the phenomenon of the evolution of sinusoidal wave into cnoidal wave, in the model execution at the an uneven bottom, before the wave enters the water with slopping bottom, the wave is given evolution zone, i.e. in front of the water in the form of water with constant depth.

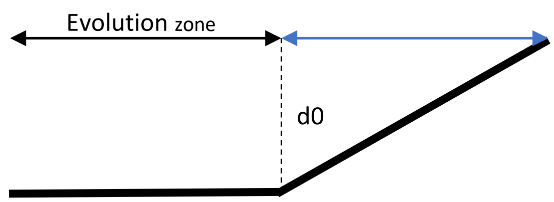

Fig.3. Sea bed for model execution at uneven bottom.

The calculation is done with evolution zone length of 100 m (Fig.3) with constant water depth of$d = 11.0\,m$, then the water depth changes until the depth of 1.0 m, with a distance of 200 m, with tangent of bottom slope i.e. $\frac{dh}{dx} = \frac{10}{200} = 0.05$. The wave used here is wave with wave period 8 sec., wave amplitude 0.794 m, with the result of calculation shown in Fig.4. and Fig.5.Coming out of the evolution zone, shoaling occurs followed by breaking, with a breaker height $H_b = 1.546$ m, at breaker depth $h_b = 1.969\,m$, where $\frac{H_b}{h_b} = 0.785$.

This condition is obtained by multiplying the second term of the water wave surface equation (15) with a factor of 2.5, so that (15) becomes,

$\left(\gamma+\frac{1}{2}\right)\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u_\eta(d+\eta)}{1.7\pi}\right) - \frac{2.5u_\eta}{2\gamma}\frac{\partial\eta}{\partial x}$.

This coefficient 2.5 is obtained by experimentation in order to obtain $\frac{H_b}{h_b}$approximates 0.80. Coefficient 2.52 can also be used where $\frac{H_b}{h_b} = 0.81$was obtained but the equation becomes unstable after the breaking. Therefore, further research is still needed on water wave surface equation as well as momentum equation that was used.

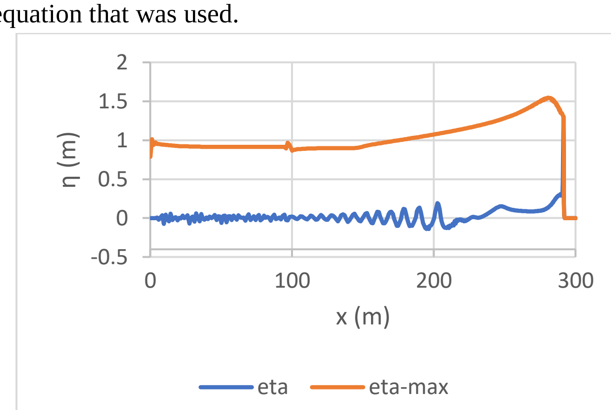

Fig 4. Wave profile (η) and wave crest elevation(η_{max})

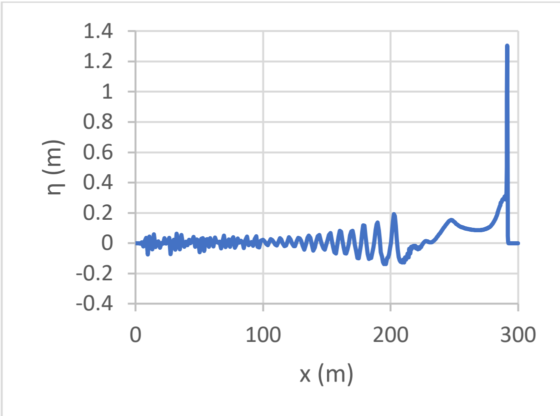

Fig 5. Wave profile (η) after breaking

VI. CONCLUSION

As has been shown that model can produce two main phenomena that occur at the water wave on its way to shallow water, i.e. shoaling and breaking. At the deep water, at a constant depth the profile of perfect cnoidal wave is formed which is also called solitary wave. Behind the main wave, wave ripple is formed which is also known as undular wave. Therefore, it can be said that the equation that was produced in this research can model several phenomena at water wave found in the nature.

Further research that needs to be done is to study the phenomenon at the equation by producing analytical solution. Considering the simple form of the equation, the analytical solution of the water wave surface equation can be obtained easily, i.e. using velocity potential equation from Laplace solution equation. By studying analytical solution, it is expected that an explanation will be obtained on the appearance of coefficient 2.5 at the second term of the water wave surface equation.

REFERENCES

[1] Boussinesq, J. (1871) Theorie de l'intumescence liquede,aplelee onde ou de translation, se propageant dans un canal rectangulaire. Comptes Rendus de l'Academic des Sciences. **72**:755-759.

[2] Dingermans, M.W. (1997). Wave propagation over uneven bottom. Advances series on Ocean Engineering **13.** World Scientific, Singapore. ISBN 978-981-02-0427-3. Archived from the original on 2012-0-08. Retrieved 2008-01-21, Chapter 5.

[3] Hamm, L.; Madsen, P.A.; Peregrine, D.A. (1993). Wave transformation in the nearshore zone: A Review. Coastal Engineering. **21** (1-3):5-39. Doi:10.1016/0378-3839(93)90044-9.

[4] Johnson, R.S. (1997). A modern introduction to the mathematical theory ofwater waves. Cambridge Texts in Applied Mathematics. **19.** Cambridge University Press ISBN 0 521 59832 X.

[5] Kirby, J.T. (2003). Boussinesq Models and Applications to nearshore wave propagation, surfzone process and waves induced currents. In Lakhan, V.C. (ed). Advances in Coastal Modeling. Elsevier Oceanography Series.**67.** Elsevier, pp. 1-41. ISBN 0 444 51149 0.

[6] Peregrine, D.H. (1967). Long waves on a Beache. Journal of Fluid Mechanics. **27** (4): 815-824. Bibcode: 1967 JFM.....815P. doi:10.1017/S0022112067002605.

[7] Peregrine, D.H. (1972). Equations for water waves and the propagation approximations behind them. In Meyer, R.E.(ed). Wave on Beaches and Resulting Sediment Transport. Academic Press . pp. 95-122. ISBN 0 12 493 250 9.

[8] Hutahaean , S. (2019a). Application of Weighted Total Acceleration Equation on Wavelength Calculation. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-2, Feb-2019. ISSN-2349-6495(P)/2456-1908(O). https://dx.doi.org/10.22161/ijaers.6.2.31

[9] Hutahaean , S. (2019b). Correlation of Weighting Coefficient at Weighted Total Acceleration With Rayleigh Distribution and with Pierson-Moskowitz Spectrum. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-3, Mar-2019. ISSN-2349-6495(P)/2456-1908(O). https://dx.doi.org/10.22161/ijaers.6.3.31

[10] Dean, R.G., Dalrymple, R.A. (1991). Water wave mechanics for engineers and scientists. Advance Series on Ocean Engineering.2. Singapore: World Scientific. ISBN 978-981-02-0420-4. OCLC 22907242.

[11] Protter, Murray, H.; Morrey, Charles, B. Jr. (1985). Differentiation Under The Integral Sign. Intermediate Calculus (second ed.). New York: Springer pp. 421-426. ISBN 978-0-387-96058-6.

[12] Anderson, John D. (1995). Computational Fluid Dynamics. The Basics With Application. ISBN 0-07-113210-4

[13] Abramowitz, M. and Stegun, I.A. (eds. 1972)."Integration". @ 25.4 in, Handbook of Mathematical Function With Formula, Graphs and Mathematical Tables, 9th printing. New York : Dover, pp. 885-887, 1972

[14] Hutahaean , S. (2019c). Wave Profile at Breaker Point. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-6, June-2019. ISSN-2349-6495(P)/2456-1908(O). https://dx.doi.org/10.22161/ijaers.6.7.59