Black Holes: a Different Perspective

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Abstract—*In this paper we propose a full revised version of a simple model, which allows a formal derivation of an infinite set of Schwarzschild-Like solutions (non-rotating and non-charged "black holes"), without resorting to General Relativity. A new meaning is assigned to the usual Schwarzschild-Like solutions (Hilbert, Droste, Brillouin, Schwarzschild), as well as to the very concepts of "black hole" and "event horizon". We hypothesize a closed Universe, homogeneous and isotropic, characterized by a further spatial dimension. Although the Universe is postulated as belonging to the so-called oscillatory class (in detail, we consider a simple-harmonically oscillating Universe), the metric variation of distances is not thought to be a real phenomenon (otherwise, we would not be able to derive any static solution): on this subject, the cosmological redshift is regarded as being caused by a variation over time of the Planck "constant". Time is considered as being absolute. The influence of matter/energy on space is analysed by the superposition of three three-dimensional scenarios. A short section is dedicated to the so-called gravitational redshift which, once having imposed the conservation of energy, may be ascribable to a local variability of the Planck "constant".*

Keywords—*Black Holes, Schwarzschild, Hilbert, Droste, Brillouin, Extra Dimension, Weak Field, Redshift.*

I. INTRODUCTION

We hypothesize a closed Universe, homogeneous and isotropic, belonging to the so-called *Oscillatory Class* [1]. The existence of a further spatial dimension is postulated. Although space, as we are allowed to perceive it, is curved, since it can be approximately identified with a Hyper-Sphere (the radius of which depends on the state of motion) [2], the Universe in its entirety, assimilated to a Four-Dimensional Ball, is to be considered as being flat. All the points are replaced by straight line segments [3] [4]: in other terms, what we perceive as being a point is actually a straight-line segment crossing the centre of the 4-Ball. Consequently, matter is not to be regarded as evenly spread on the (Hyper)Surface of the 4-Ball, but rather as homogeneously filling the 4-Ball in its entirety. We have elsewhere [4] deduced the following identity:

$$R_m = \frac{2GM_{tot,m}}{c^2} \qquad (1)$$

G represents the *Gravitational Constant*, c the *Speed of Light*, R_m the mean value of the radius of the 4-ball, and $M_{tot,m}$ the corresponding mass. According to our model, R_m and $M_{tot,m}$ can be conventionally considered as being real values, since the metric variation of the cosmological distances is not thought to be a real phenomenon (in other terms, we hypothesize that the real amount of space between whatever couple of points remains constant with the passing of time) [4] [5]. In this regard, we specify how, in order to legitimize the so-called *Cosmological Redshift*, the *Plank Constant* may vary over time [6] [7].

Replacing, for convenience, $M_{tot,m}$ with M_{tot}, and R_m with Rs (the *Schwarzschild Radius*), from *(1)* we have:

$$R_s = \frac{2GM_{tot}}{c^2} \qquad (2)$$

The Universe we have hypothesized may be approximately described, with obvious meaning of the notation, by the following inequality:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R_s^2 \qquad (3)$$

The Universe we are allowed to perceive (static configuration) can be assimilated to the Hyper-Surface defined by the underlying identity:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R_s^2 \qquad (4)$$

Let us denote with C the centre of the 4-ball, with O and P two points on the surface, the first of which taken as origin, and with O' the centre of the so-called *Measured Circumference*, to which P belongs. Both O and O' are considered as belonging to x_4. The *Angular Distance* between O and P, as perceived by an ideal observer placed in C, is denoted by χ.

The arc bordered by O and P, denoted by R_p, represents the so-called *Proper Radius* (the measured distance between the above-mentioned points). We have:

$$R_p(\chi) = R_s\chi \qquad (5)$$

The straight-line segment bordered by O' and P, denoted by R_c, represents the so-called *Predicted (or Forecast) Radius* (the ratio between the perimeter of the Measured Circumference and 2π). We have:

$$R_c(\chi) = X(\chi) = R_s \sin \chi \qquad (6)$$

From the previous we immediately deduce:

$$\chi = \arcsin\left(\frac{X}{R_s}\right) \qquad (7)$$

Consequently, we have:

$$dR_p = R_s d\chi = \frac{dX}{\sqrt{1 - \left(\frac{X}{R_s}\right)^2}} \qquad (8)$$

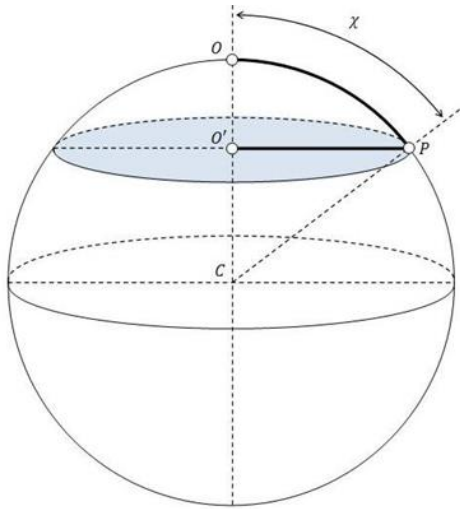The scenario is qualitative depicted in *Figure 1*.

Figure 1. 4-Ball

At this point, for the Hyper-Surface defined in *(4)*, the *Friedmann–Robertson–Walker* metric [8] can be written:

$$ds^2 = c^2 dt^2 - \frac{dX^2}{1 - \left(\frac{X}{R_s}\right)^2} - X^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (9)$$

Let us denote with S_2 the 2-Sphere characterized by a radius of curvature equal to R_c. In order to simplify the notation, from now onwards we shall denote with the same symbol both the geometrical object and the corresponding surface area or volume. Consequently, we have:

$$S_2(\chi) = X^2 \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta d\theta d\varphi = X^2 \int_0^{4\pi} d\Omega = 4\pi X^2 \qquad (10)$$
$$= 4\pi R_s^2 \sin^2\chi$$

The above-mentioned surface is simultaneously border of a 3-Ball, denoted by V_3, and of a Hyper-Spherical Cap, denoted by S_3. V_3 represents the *Predicted (or Forecast) Volume*, S_3 the *Proper Volume*. We have:

$$V_3(\chi) = \int_0^{R_c} S_2(\chi)dR_c = 4\pi \int_0^X X^2 dX = \frac{4}{3}\pi X^3 \qquad (11)$$
$$= \frac{4}{3}\pi R_s^2 \sin^3\chi$$

$$S_3(\chi) = \int_0^{R_p} S_2(\chi)\, dR_p = 4\pi R_s^3 \int_0^\chi \sin^2\chi \, d\chi \qquad (12)$$
$$= 2\pi R_s^3(\chi - \sin\chi\cos\chi)$$

We can generalize the foregoing as follows:

$$S_3(R,\chi) = 2\pi r^3(\chi - \sin\chi\cos\chi) \qquad R \in [0, R_s] \qquad (13)$$

The Hyper-Surface S_3 defined in *(12)* is associated to a Hyper-Spherical Sector, denoted by V_4. We have:

$$V_4(\chi) = \int_0^{R_s} S_3(R,\chi)dR = 2\pi(\chi - \sin\chi\cos\chi)\int_0^{R_s} R^3 dR \qquad (14)$$
$$= \frac{1}{2}\pi R_s^4(\chi - \sin\chi\cos\chi)$$

II.   GRAVITY: HOW MASS "BENDS" SPACE
1. Gravitational "Singularities"
As previously stated, the (curved) space we are allowed to perceive can be approximately identified with a Hyper-Sphere, the radius of which depends on our state of motion: at rest, this radius equates R_s. In our simple model the total amount of mass is constant: in other terms, mass can only be redistributed. Let us consider a generic point Q, belonging to the surface of the 4-Ball, and let us denote with χ_{max} the angular distance between this point and the origin O. In order to create a "gravitational singularity" in correspondence of the origin, we have to ideally concentrate in O, from the point of view of an observer at rest (who is exclusively allowed to perceive a three-dimensional curved universe), all the mass enclosed in the 2-Sphere defined by *(10)* (with $\chi=\chi_{max}$). This surface represents the border of the Hyper-Spherical Cap defined in *(12)* (with $\chi=\chi_{max}$) which, in turn, is associated to the hyper-spherical sector defined by *(14)* (with $\chi=\chi_{max}$).
According to our theory, in enacting the ideal procedure previously expounded, we actually hypothesize that all the mass of the Hyper-Spherical Sector earlier defined may be concentrated (and evenly spread) along the material segment bordered by C and O. The procedure entails a linear mass (energy) density increment, no longer compatible with the previous radial extension: consequently, both the segment and the corresponding space undergo a radial contraction (the segment shortens together with space) and the surrounding spatial lattice, the integrity of which must be in any case preserved, results deformed. We want to determine the new radial extension of the segment (that represents the singularity) and the shape of the deformed spatial lattice.
It is worth specifying how, abiding to the global symmetry elsewhere introduced [2] [4] and herein taken for granted, the procedure previously exploited is symmetric with respect to the centre of the 4-Ball: consequently, we should have actually considered two opposite Hyper-Spherical Sectors, characterized by the same amplitude, and a single material segment, crossing the centre C, bordered by O and its antipodal point.

2. Three-Dimensional Scenarios
From *(3)*, by setting equal to zero, one at a time, x_1, x_2 and x_3, we obtain the following three-dimensional scenarios:

$$x_{4,1}^2 + x_2^2 + x_3^2 \leq R_s^2 \qquad (15)$$

$$x_1^2 + x_{4,2}^2 + x_3^2 \leq R_s^2 \qquad (16)$$

$$x_1^2 + x_2^2 + x_{4,3}^2 \leq R_s^2 \qquad (17)$$

Evidently, if we take into consideration one among the static scenarios we have just obtained, the procedure previously discussed (the creation of the singularity) is equivalent to concentrating along a segment the mass of a spherical sector.

Let us denote with S_{2-1}, the Circumference defined by the following relation:

$$S_{2-1}(\chi) = 2\pi X \qquad (18)$$

In the three dimensional scenario we have been considering, S_{2-1} "plays the role" of S_2, defined in *(10)*.

The circumference defined in *(18)* is simultaneously border of a Disc, denoted by V_{3-1}, and of a Sphere, denoted by S_{3-1}. In the three-dimensional scenario we have been considering, the first "plays the role" of the Predicted (or Forecast) Volume V_3, defined in *(11)*, while the second "plays the role" of the Proper Volume S_3, defined in *(12)*. We have:

$$V_{3-1}(\chi) = \int_0^{R_c} S_{2-1}(\chi) dR_c \qquad (19)$$
$$= 2\pi \int_0^X X dX = \pi X^2 = \pi R_s^2 \sin^2 \chi$$

$$S_{3-1}(\chi) = \int_0^{R_p} S_{2-1}(\chi) \, dR_p = 2\pi \int_0^\chi \sin\chi \, d\chi \qquad (20)$$
$$= 2\pi R_s^2 (1 - \cos\chi)$$

We can generalize the foregoing as follows:

$$S_{3-1}(R,\chi) = 2\pi r^2 (1 - \cos\chi) \qquad R \in [0, R_s] \qquad (21)$$

Consequently, S_{3-1} is associated to a Spherical Sector, denoted by V_{4-1}, characterized by a volume provided by the following relation:

$$V_{4-1}(\chi) = \int_0^{R_s} S_{3-1}(R,\chi) dR = 2\pi(1 - \cos\chi) \int_0^{R_s} R^2 dR \qquad (22)$$
$$= \frac{2}{3}\pi R_s^3 (1 - \cos\chi)$$

In the three dimensional scenario we have been considering, V_{4-1} "plays the role" of V_4, defined in *(14)*.

As previously highlighted, the new radial extension of the segment (that represents the singularity) is still unknown, as well as the shape of the deformed spatial lattice. Let us carry out some hypotheses.

Let us denote with r the *Radial Coordinate* of a generic point of the warped surface. Now, let's suppose that, notwithstanding the deformation of the spatial lattice induced by the mass, if the angular distance between whatever couple of points does not vary, the corresponding

measured distance remains constant. Actually, there is no point in hypothesizing a different behaviour.

From now onwards, we shall resort to the subscript "g" every time we refer to a quantity measured after the creation of the singularity.

We must impose the following:

$$R_{p,g} = \int_0^\chi \sqrt{\left(\frac{dr}{d\chi}\right)^2 + r^2} \, d\chi = R_s \chi = R_p \qquad (23)$$

$$R_s^2 = \left(\frac{dr}{d\chi}\right)^2 + r^2 \qquad (24)$$

From the previous we easily obtain the following banal differential equation:

$$\frac{d^2 r}{d\chi^2} + r = 0 \qquad (25)$$

The boundary conditions can be easily determined by resorting to the well-known shell theorem: in other terms, we have to impose that, for all the points belonging to the circumference defined in *(18)* once having set $\chi = \chi_{max}$ (actually, for all the points belonging to the 2-Sphere defined in *(10)*, once having set $\chi = \chi_{max}$), there must be no difference between the initial condition and the final one (matter concentrated in a single point).

Therefore, we have:

$$\frac{dr}{d\chi}(\chi_{max}) = 0 \qquad (26)$$

$$r(\chi_{max}) = R_s \qquad (27)$$

From *(25)*, taking into account *(26)* and *(27)*, we obtain:

$$r(\chi) = R_s \cos(\chi_{max} - \chi) \qquad (28)$$

From the previous, we can immediately deduce:

$$r_{min} = r(0) = R_s \cos\chi_{max} \qquad (29)$$
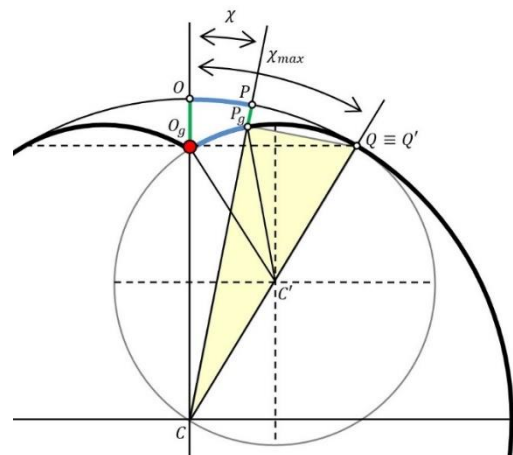
The scenario is qualitative depicted in *Figure 2*

Figure 2. Gravitational Singularity

Figure 2 qualitatively shows how space results in being deformed due to the Gravitational Singularity, perceived as being placed in O_g. At the beginning, the origin coincides

with O. If we concentrate in O (actually along the segment bordered by C and O) the mass of the Spherical Sector (actually a Hyper-Spherical Sector) with an amplitude equal to $2\chi_{max}$, space undergoes a contraction. The new origin coincides with O_g, and the surrounding space is symmetrically warped. The initial radial coordinate of a generic point P (actually its initial radial extension) is represented by the segment bordered by C and P. The corresponding angular distance is denoted by χ. The final coordinate (actually the final radial extension), represented by the segment bordered by C and P_g, is shorter than the initial one, and its value is provided by *(28)*. The proper radius does not undergo any modification: the arc bordered by O and P, in fact, is evidently equal to the one bordered by O_g and Pg.

If we denote with x the *Reduced "Flat" Coordinate* (the *Reduced Forecast Radius*), we have:

$$R_{c,g} = x = r \sin \chi = R_s \sin \chi \cos(\chi_{max} - \chi) \quad (30)$$

Moreover, with obvious meaning of the notation, we can immediately write:

$$\delta(\chi) = R_s - r(\chi) = R_s[1 - \cos(\chi_{max} - \chi)] \quad (31)$$

$$\delta_{max} = \delta(0) = R_s(1 - \cos \chi_{max}) \quad (32)$$

If we denote with M_{tot} the mass of the Ball (that "plays the role" of the 4-Ball with which we identify our Universe), and with $M_{\chi,max}$ the mass contained in the spherical sector with an amplitude equal to $2\chi_{max}$ (which, as previously remarked, "plays the role" of a Hyper-Spherical Sector), we can write, taking into account *(32)*, the following:

$$\frac{M_{\chi max}}{M_{tot}} = 1 - \cos \chi_{max} = \frac{\delta_{max}}{R_s} \quad (33)$$

$$\delta_{max} = R_s \frac{M_{\chi max}}{M_{tot}} = \frac{2GM_{\chi max}}{c^2} = R_{s,\chi max} \quad (34)$$

In other terms, the procedure entails a reduction of the radial coordinate of O (actually, the material segment bordered by C and O undergoes a contraction) the size of which is equal to the Schwarzschild radius of $M_{\chi max}$.

The scenario is qualitatively portrayed in the following figure, where the singularity (as we can perceive it) coincides with the point O_g.
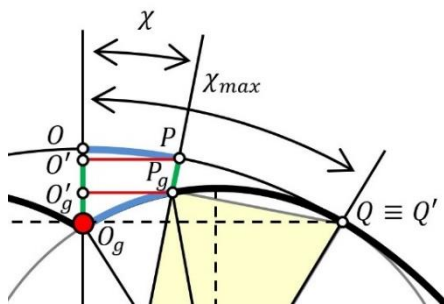

Figure 3. Gravitational Singularity (Particular)

Figure 3 shows once again how the singularity, perceived as being placed in O_g, does not influence the measured distance (the proper radius). The arc bordered by O and P, as previously underlined, is evidently equal to the one bordered by O_g and P_g. On the contrary, the "Flat" Coordinate (the Forecast Radius) undergoes a reduction. The segment bordered by B and P represents the Forecast Radius (X) when matter is evenly spread; the segment bordered by B_g and P_g represents the Reduced Forecast Radius (x).

III. QUANTIZATION

If mass homogeneously fills the 4-Ball with which we identify the Universe (static configuration), by virtue of the symmetry [3] [4], the *Energy* of a *Material Segment*, provided with a mass M, can be written as follows:

$$E = Mc^2 \quad (35)$$

The *Linear Mass Density* [3] [4] is defined as follows:

$$\bar{M} = \frac{M}{R_s} \quad (36)$$

By virtue of the foregoing, the *(Linear) Energy Density* can be defined as follows:

$$\bar{E} = \frac{E}{R_s} = \frac{Mc^2}{R_s} = \bar{M}c^2 \quad (37)$$

If we denote with ΔR_m the *(Radial) Quantum of Space* [4], the *Punctual Mass*, denoted by m, is defined as follows:

$$m = \bar{M}\Delta R_{min} = \frac{M}{R_s}\Delta R_{min} \quad (38)$$

As for the corresponding *Energy*, by virtue of *(37)* and *(38)*, we can immediately write:

$$E_m = \bar{E}\Delta R_{min} = \frac{Mc^2}{R_s}\Delta R_{min} = mc^2 \quad (39)$$

Let us denote with M_{min} the *Minimum Linear Mass*. The corresponding *Energy* can be obviously written as follows:

$$E_{min} = M_{min}c^2 \quad (40)$$

As for the *Minimum Linear Mass Density* we have:

$$\bar{M}_{min} = \frac{M_{min}}{R_s} \quad (41)$$

The *Minimum (Linear) Energy Density* is clearly provided by the following:

$$\bar{E}_{min} = \frac{E_{min}}{R_s} = \frac{M_{min}c^2}{R_s} = \bar{M}_{min}c^2 \quad (42)$$

The *Minimum Punctual Mass*, denoted by m_{min}, is defined as follows:

$$m_{min} = \bar{M}_{min}\Delta R_{min} = \frac{M_{min}}{R_s}\Delta R_{min} \quad (43)$$

Consequently, as for the *Energy* related to the above-mentioned mass, we have:

$$E_{m,min} = \bar{E}_{min}\Delta R_{min} = \frac{M_{min}c^2}{R_s}\Delta R_{min} = m_{min}c^2 \quad (44)$$

By virtue of *(34)*, we can write the expression for the *Minimum Schwarzschild Radius*:

$$R_{s,min} = \frac{2GM_{min}}{c^2} \qquad (45)$$

Now, taking into account the symmetry, the *Maximum Wavelength* for a photon can be written as follows:

$$\lambda_{max} = \pi R_s \qquad (46)$$

Denoting with *h*, as usual, the Planck Constant, we can determine the *Minimum (Perceived) Energy*:

$$E_{photon,min} = \frac{hc}{\lambda_{max}} \qquad (47)$$

From *(44)* and *(47)*, we can easily obtain the expression for the Minimum Punctual Mass:

$$E_{photon,min} = \frac{hc}{\lambda_{max}} = \frac{hc}{\pi R_s} = m_{min}c^2 \qquad (48)$$

$$m_{min} = \frac{h}{\pi c R_s} \qquad (49)$$

For a (linear) mass to induce a spatial deformation (a radial contraction), the value of the corresponding Schwarzschild Radius must be greater than or equal to the value of the (Radial) Quantum of Space.
Consequently, we have:

$$R_{s,\chi_{max}} \geq \Delta R_{min} \qquad (50)$$

If we banally impose that M_{min} represents the value of linear mass, still unknown, below which no deformation of spatial lattice (no radial contraction) occurs, we can carry out the following (upper-limit) position:

$$R_{s,min} = \Delta R_{min} \qquad (51)$$

When mass homogeneously fills the 4-Ball, denoting with $\mathcal{N}$ an integer (the *Number of Radial Quanta*), we have:

$$R_s = \mathcal{N}\Delta R_{min} \qquad (52)$$

Now, from *(43)*, *(45)* and *(51)* we have:

$$m_{min} = \frac{M_{min}}{R_s}\Delta R_{min} = \frac{M_{min}}{R_s}R_{s,min} = \frac{2GM_{min}^2}{R_s c^2} \qquad (53)$$

From the previous, by virtue of *(49)*, we obtain:

$$\frac{2GM_{min}^2}{R_s c^2} = \frac{h}{\pi c R_s} \qquad (54)$$

From the previous, taking into account the definition of *Reduced Planck Constant*, we finally obtain:

$$M_{min}^2 = \left(\frac{h}{2\pi}\right)\frac{c}{G} = \frac{\hbar c}{G} \qquad (55)$$

$$M_{min} = \sqrt{\frac{\hbar c}{G}} = M_P \qquad (56)$$

The previous represents the Minimum Value for Linear Mass. It is worth underlining how this value formally coincides with the one of the so-called *Planck Mass*, herein denoted with M_p.

From *(56)*, taking into account *(43)* and *(52)*, for the Minimum Punctual Mass we have:

$$m_{min} = \frac{\Delta R_{min}}{R_s}M_{min} = \frac{1}{\mathcal{N}}M_{min} = \frac{1}{\mathcal{N}}\sqrt{\frac{\hbar c}{G}} \qquad (57)$$

Finally, from *(45)* and *(56)*, we obtain the value of the (Radial) Quantum of Space:

$$R_{s,min} = \frac{2GM_{min}}{c^2} = 2\sqrt{\frac{\hbar G}{c^3}} = 2\ell_P = \Delta R_{min} \qquad (58)$$

At this point, we can also carry out a Time Quantization. Taking into account the previous, denoting with t_p the so-called *Planck Time*, we define the *Quantum of Time* as follows:

$$\Delta t_{min} = \frac{\Delta R_{min}}{c} = \frac{\Delta r_{min,\pi/2}}{c} = \Delta t_{min,\pi/2} = 2\sqrt{\frac{\hbar G}{c^5}} = 2t_P \qquad (59)$$

We can now start concretely building our simple model of (non-rotating and non-charged) *"Black Hole"*.

IV. "BLACK HOLES"

1. Short Introduction

Let us suppose that the total available mass may be concentrated in *O*. Abiding by our model, from *(27)* and *(28)*, by setting $\chi_{max}=\pi/2$, we can write the following:

$$r(\chi) = R_s \cos\left(\frac{\pi}{2} - \chi\right) = R_s \sin\chi \qquad (60)$$

$$r_{max} = r\left(\frac{\pi}{2}\right) = R_s \qquad (61)$$

Evidently, the value of the Radial Coordinate (the *Reduced Radial Extension*) coincides, for any χ, with the one of the Predicted Radius provided by *(6)*:

$$R_c = X = r \qquad (62)$$

For the Reduced Predicted Radius, we have:

$$R_{c,g} = R_c \sin\chi \qquad (63)$$

$$R_{c,g} = x = X\sin\chi = R_s \sin^2\chi \qquad (64)$$

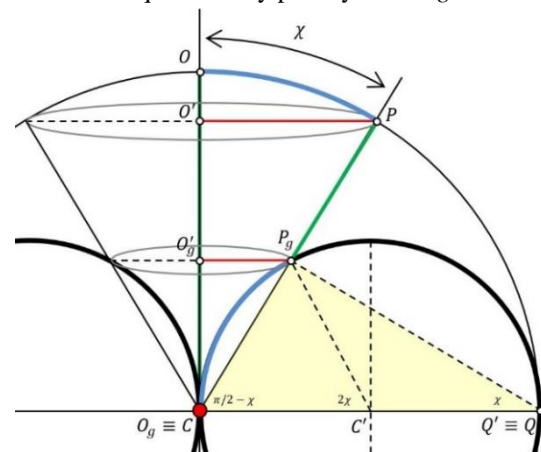The scenario is qualitatively portrayed in *Figure 4*.

Figure 4. "Black Hole"

As for S_2, V_3 and S_3, the Singularity induces the following modifications:

$$S_{2,g}(\chi) = 4\pi x^2 = 4\pi R_s^2 \sin^4 \chi \qquad (65)$$

$$V_{3,g}(\chi) = \int_0^{R_{c,g}} S_{2,g}(\chi) dR_{c,g} = 4\pi \int_0^x x^2 dx = \frac{4}{3}\pi x^3 \qquad (66)$$

$$= \frac{4}{3}\pi R_s^3 \sin^6 \chi$$

$$S_{3,g}(\chi) = \int_0^{R_p} S_{2,g}(\chi) dR_p = 4\pi R_s^3 \int_0^\chi \sin^4 \chi \, d\chi$$

$$= \frac{\pi}{2} R_s^3 (\chi - \sin\chi \cos\chi + \qquad (67)$$

$$- 2\sin^3\chi \cos\chi)$$

2. Variable Space-Quantum

We want to carry out a quantization of the coordinate *r*. As shown in *(60)*, this coordinate depends on the angular distance χ: the more we approach the "Singularity", the more the value of *r* decreases.

However, once again, *r* does not shorten within space: it shortens together with space, since space itself undergoes a progressive (radial) contraction in approaching the "singularity".

Consequently, we consider a *Variable (Radial) Space-Quantum*, denoted with $\Delta r_{\chi,min}$, the value of which depends on the angular distance χ.

If $\mathcal{N}$ represents the same integer introduced in *(52)*, we impose the following:

$$r = \mathcal{N}\Delta r_{min,\chi} \qquad (68)$$

According to the previous, taking into account *(52)* and *(61)*, we must have:

$$r_{max} = \mathcal{N}\Delta r_{min,\pi/2} = R_s = \mathcal{N}\Delta R_{min} \qquad (69)$$

Consequently, by virtue of *(58)*, we can write:

$$\Delta r_{min,\pi/2} = \Delta R_{min} = 2\ell_P \qquad (70)$$

From *(68)* and *(69)* we immediately obtain:

$$\mathcal{N} = \frac{r}{\Delta r_{min,\chi}} = \frac{R_s}{\Delta R_{min}} = \frac{R_s}{\Delta r_{min,\pi/2}} \qquad (71)$$

From the foregoing, taking into account *(60)*, we have:

$$\frac{\Delta r_{min,\chi}}{\Delta r_{min,\pi/2}} = \frac{\Delta r_{min,\chi}}{\Delta R_{min}} = \frac{r}{R_s} = \sin\chi \qquad (72)$$

In the light of the previous relation, we can now introduce the following *Non-Dimensional Parameter*, which represents nothing but a simple *Scale Factor*:

$$\eta_{\Delta r_{min}} = \frac{\Delta r_{min,\pi/2}}{\Delta r_{min,\chi}} = \frac{\Delta R_{min}}{\Delta r_{min,\chi}} = \frac{R_s}{r} = \frac{1}{\sin\chi} \qquad (73)$$

Now, from *(58)* and *(72)*, we immediately obtain:

$$\Delta r_{min,\chi} = \sin\chi \Delta R_{min} = 2\sin\chi \sqrt{\frac{\hbar G}{c^3}} = 2\sqrt{\frac{\hbar_\chi G}{c^3}} = 2\ell_{P,\chi} \qquad (74)$$

In other terms, we have been hypothesizing a local variability of the Planck "Constant". From the previous, taking into account *(60)*, we easily deduce the following:

$$\frac{h_\chi}{h} = \left(\frac{r}{R_s}\right)^2 = \sin^2\chi = \frac{h_\chi}{h_{\pi/2}} \qquad (75)$$

The *Variable Quantum of Time* is defined as follows:

$$\Delta t_{min,g}(\chi) = \Delta t_{min,\chi} = \frac{\Delta r_{min,\chi}}{c} = 2\sqrt{\frac{\hbar_\chi G}{c^5}} = 2t_{P,\chi} \qquad (76)$$

By virtue of *(59)* and *(73)*, from the previous we obtain:

$$\frac{\Delta t_{min,\chi}}{\Delta t_{min,\pi/2}} = \frac{\Delta t_{min,\chi}}{\Delta t_{min}} = \sin\chi = \frac{1}{\eta_{\Delta r_{min}}} \qquad (77)$$

3. "Gravitational" Mass

In case of singularity, a material segment does not undergo any radial reduction (in other terms, it does not shorten within space): as previously remarked, both the segment and the corresponding space undergo a radial contraction (the segment shortens together with space).

Consequently, if we denote with *M* the Mass of a *"Test" Material Segment*, the *(Variable) Linear Mass Density*, in case of gravitational singularity, con be defined as follows:

$$\bar{M} = \frac{M}{r} \qquad (78)$$

As for the Mass of a *Test Particle* (the mass we perceive), by virtue of *(71)* and *(78)*, we can write, with obvious meaning of the notation, the following:

$$m_\chi = \bar{M}\Delta r_{min,\chi} = M\frac{\Delta r_{min,\chi}}{r} = M\frac{\Delta r_{min,\pi/2}}{R_s} = m_{\pi/2} \qquad (79)$$

From the previous, by virtue of *(38)* and *(52)*, we have:

$$m_\chi = m_{\pi/2} = \frac{M}{R_s}\Delta R_{min} = \frac{M}{\mathcal{N}} = m \qquad (80)$$

In other terms, thanks to the position in *(68)* (the meaning of which should now be clearer), the *"Gravitational" Mass* and the inertial one coincide (as requested by the *Equivalence Principle*) [9].

4. Conservation of Energy

As elsewhere deduced, the *Conservation of Energy* for a *Free Material Segment* can be written as follows [3] [4]:

$$E = Mc^2 = M_r v^2 + \left(\frac{r}{R_s}\right)^2 M_r c^2 + (M - M_r)c^2 \qquad (81)$$

In our case, by virtue of what has been specified in the previous paragraph, bearing in mind the meaning of *r*, we have to banally impose:

$$M = M_r \qquad (82)$$

As a consequence, for a Test Material Segment, the motion of which is induced by a *(Gravitational) Potential*, from *(81)* and *(82)* we immediately obtain:

$$E = Mc^2 = Mv^2 + \left(\frac{r}{R_s}\right)^2 Mc^2 \qquad (83)$$

From the previous relation, taking into account *(80)*, we immediately obtain the *Conservation of Energy for a (Free-Falling) Test Particle*:

$$E_m = \frac{E}{\mathcal{N}} = \frac{M}{\mathcal{N}}c^2 = \frac{M}{\mathcal{N}}v^2 + \left(\frac{r}{R_s}\right)^2 \frac{M}{\mathcal{N}}c^2 \tag{84}$$

$$E_m = mc^2 = mv^2 + \left(\frac{r}{R_s}\right)^2 mc^2 \tag{85}$$

5. The (Gravitational) Potential and the Coordinate R*

From *(60)* and *(85)* we can easily deduce:

$$v = c\sqrt{1 - \left(\frac{r}{R_s}\right)^2} = c\cos\chi \tag{86}$$

$$\frac{r}{R_s} = \sqrt{1 - \left(\frac{v}{c}\right)^2} = \sin\chi \tag{87}$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mc^2\left[1 - \left(\frac{r}{R_s}\right)^2\right] = \frac{1}{2}mv^2 - \frac{1}{2}mc^2\cos^2\chi = 0 \tag{88}$$

From *(2)* we immediately obtain:

$$c^2 = \frac{2GM_{tot}}{R_s} \tag{89}$$

Consequently, we have:

$$-\frac{1}{2}c^2\cos^2\chi = -\frac{GM_{tot}}{R_s}\cos^2\chi \tag{90}$$

Let us introduce a *New Coordinate* [10], denoted by R^*, defined as follows:

$$R^*(\chi) = \frac{R_s}{\cos^2\chi} \tag{91}$$

Obviously, from the previous we have:

$$R^*(0) = R_s \tag{92}$$

$$\lim_{\chi\to\pi/2} R^* = +\infty \tag{93}$$

$$\cos\chi = \sqrt{\frac{R_s}{R^*}} \tag{94}$$

$$\sin\chi = \sqrt{1 - \frac{R_s}{R^*}} \tag{95}$$

From (), taking into account (), we obtain:

$$\frac{1}{2}mv^2 - \frac{GM_{tot}}{R^*}m = 0 \tag{96}$$

Let us define the *Pseudo-Newtonian Potential*, denoted by φ, as follows:

$$-\frac{GM_{tot}}{R^*} = -\frac{1}{2}c^2\cos^2\chi = \phi(\chi) \tag{97}$$

Evidently, with obvious meaning of the notation, from the previous we have:

$$R^* = R_s \Rightarrow \phi(\chi) = \phi(0) = \phi_{min} = -\frac{1}{2}c^2 \tag{98}$$

From (), taking into account (), we immediately obtain:

$$\frac{1}{2}mv^2 + \phi m = 0 \tag{99}$$

6. Speed of a Free-Falling Particle

From *(2)*, *(97)* and *(99)*, we have:

$$v = \sqrt{2\phi} = \sqrt{\frac{2GM_{tot}}{R^*}} = c\sqrt{\frac{2GM_{tot}}{c^2R^*}} = c\sqrt{\frac{R_s}{R^*}} = v_{escape} \tag{100}$$

$$v = c\cos\chi \tag{101}$$

The *Speed* consists of two *Components*, denoted by v_I and v_{II}. We can evidently write:

$$v = \sqrt{v_I{}^2 + v_{II}{}^2} \tag{102}$$

From *(100)* we can easily deduce:

$$\frac{R_s}{R^*} = \frac{2\phi}{c^2} \tag{103}$$

Consequently, v_I and v_{II} assume the following forms:

$$v_I = c\sin\chi\cos\chi = c\sqrt{\left(1 - \frac{R_s}{R^*}\right)\frac{R_s}{R^*}} = c\sqrt{\frac{2\phi}{c^2} - \left(\frac{2\phi}{c^2}\right)^2} \tag{104}$$

$$v_{II} = c\cos^2\chi = c\frac{R_s}{R^*} = \frac{2\phi}{c} \tag{105}$$

The components of speed are depicted in *Figure 5*
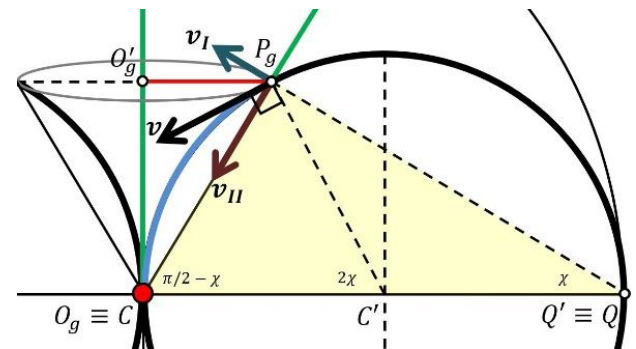

Figure 5. Speed of a Free-Falling Particle

Figure 5 shows how, when a test particle approaches the singularity, the value of v_I decreases while, on the contrary, the value of v_{II} increases. It is commonly said that, in approaching the singularity, the Space-Like Geodesics become Time-Like, and vice-versa. In our case, the above-mentioned interpretation is not correct, since the radial coordinate is nothing but the extension of a material segment, that we perceive as being a material point (the Test Particle). The straight-line segment bordered by *C* (that evidently coincides with O_g) and P_g represents the radial extension of the particle, the one bordered by O_g' and P_g represents the Reduced "Flat Coordinate" (the Radius of the *Reduced Circumference*).

7. Parameterization

We want to find two new coordinates, related to each other, that could "play the role" of R_s and *r*.
Firstly, in the light of *(23)*, we must impose:

$$\int_0^\chi \sqrt{\left(\frac{dr_K^*}{d\chi}\right)^2 + r_K^{*2}}\, d\chi = R_{p,g}^* = R_p^* \tag{106}$$

$$= \int_0^\chi \sqrt{\left(\frac{dR_K^*}{d\chi}\right)^2 + R_K^*}\, d\chi$$

Secondly, in the light of *(60)*, we must additionally impose:

$$r_K^* = R_K^* \sin \chi \tag{107}$$

From *(106)* and *(107)* we easily obtain the following:

$$\frac{dR_K^*}{R_K^*} = 2 \tan \chi \, d\chi \tag{108}$$

The general solution of the foregoing, denoting with *K* an arbitrary constant, is:

$$R_K^* = \frac{K}{\cos^2 \chi} \tag{109}$$

From the previous we immediately deduce the underlying noteworthy identity:

$$\sin \chi = \sqrt{1 - \frac{K}{R_K^*}} \tag{110}$$

From *(109)* we have:

$$\frac{dR_K^*}{d\chi} = 2K \frac{\sin \chi}{\cos^3 \chi} \tag{111}$$

From *(107)* and *(109)*, we have:

$$r_K^* = R_K^* \sin \chi = K \frac{\sin \chi}{\cos^2 \chi} \tag{112}$$

$$\frac{dr_K^*}{d\chi} = K \frac{1 + \sin^2 \chi}{\cos^3 \chi} \tag{113}$$

As for the Predicted Radius, coherently with *(62)*, we have:

$$R_c^* = X_K^* = R_K^* \sin \chi = r_K^* \tag{114}$$

In the light of *(63)*, the relation between the Predicted Radiuses with (additional subscript "*g*") and without (no additional subscript) the Singularity must be the following:

$$R_{c,g}^* = R_c^* \sin \chi \tag{115}$$

Therefore, as for the Reduced Predicted Radius we have:

$$R_{c,g}^* = x_K^* = R_c^* \sin \chi = X_K^* \sin \chi = r_K^* \sin \chi = R_K^* \sin^2 \chi \tag{116}$$

From the previous, taking into account *(109)*, we obtain:

$$x_K^* = K \tan^2 \chi = \frac{K}{\cos^2 \chi} - K = R_K^* - K \tag{117}$$

$$\frac{dx_K^*}{d\chi} = \frac{dR_K^*}{d\chi} \tag{118}$$

According to *(106)*, the Proper Radius is not influenced by the singularity. Therefore, from *(109)* and *(111)* we obtain:

$$dR_p^* = \sqrt{\left(\frac{dR_K^*}{d\chi}\right)^2 + R_K^{*2}}\, d\chi = \frac{dR_K^*}{d\chi}\sqrt{1 + \frac{1}{4\tan^2\chi}}\, d\chi$$
$$= \sqrt{1 + \frac{1}{4\tan^2\chi}}\, dR_K^* = dR_{p,g}^* \tag{119}$$

In *Figure 6* a useful comparison between old and new (Parameterized) Coordinates, once having set $K=R_s$, is qualitatively displayed.
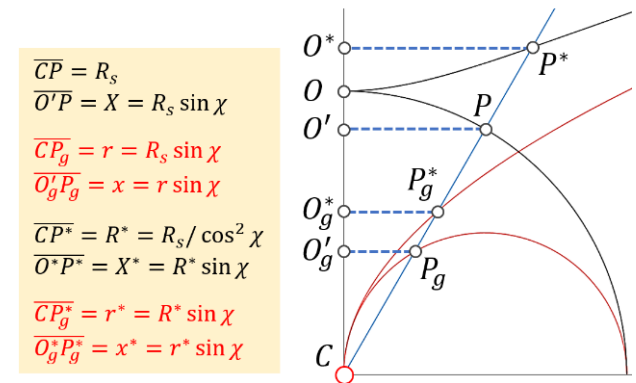

Figure 6. Parameterization ($K=R_s$)

8. Parameterized Quantization

The parameterization also affects the quantization. Obviously, it is not a real phenomenon.

Coherently with the parameterization we have been resorting to, by virtue of *(52)* we must now impose:

$$R_K^* = \mathcal{N} \Delta R_{min}^* = \mathcal{N} \Delta r_{min,\pi/2}^* \tag{120}$$

From the previous, taking into account *(109)*, we obtain:

$$\Delta R_{min}^* = \frac{R_K^*}{\mathcal{N}} = \frac{K}{\mathcal{N}} \frac{1}{\cos^2 \chi} \tag{121}$$

If we set $K=R_s$, taking into account *(52)*, the foregoing can be written as follows:

$$\Delta R_{min}^* = \frac{R_s}{\mathcal{N}} \frac{1}{\cos^2 \chi} = \frac{\Delta R_{min}}{\cos^2 \chi} \tag{122}$$

Obviously, by virtue of *(68)*, we must also impose:

$$r_K^* = \mathcal{N} \Delta r_{min,\chi}^* \tag{123}$$

From the previous, taking into account *(112)*, we obtain:

$$\Delta r_{min,\chi}^* = \frac{r_K^*}{\mathcal{N}} = \frac{K}{\mathcal{N}} \frac{\sin \chi}{\cos^2 \chi} \tag{124}$$

If we set $K=R_s$, taking into account *(68)*, the foregoing can be written as follows:

$$\Delta r_{min}^* = \frac{R_s \sin \chi}{\mathcal{N}} \frac{1}{\cos^2 \chi} = \frac{\Delta r_{min,\chi}}{\cos^2 \chi} \tag{125}$$

Evidently, by virtue of *(121)* and *(124)*, we can write:

$$\Delta r_{min,\pi/2}^* = \Delta R_{min}^* \tag{126}$$

From *(72)*, *(122)* and *(124)*, taking into account the foregoing, we have:

$$\frac{\Delta r_{min,\chi}^*}{\Delta r_{min,\pi/2}^*} = \frac{\Delta r_{min,\chi}^*}{\Delta R_{min}^*} = \frac{\Delta r_{min,\chi}}{\Delta R_{min}} = \frac{\Delta r_{min,\chi}}{\Delta r_{min,\pi/2}} = \sin \chi \tag{127}$$

In the light of the previous, resorting to *(110)*, we can now introduce the new following *Parameterized Scale Factor*:

$$\eta_{\Delta r_{min}}^* = \frac{\Delta R_{min}^*}{\Delta r_{min,\chi}^*} = \frac{\Delta R_{min}}{\Delta r_{min,\chi}} = \frac{1}{\sin \chi} = \frac{1}{\sqrt{1 - \frac{K}{R_K^*}}} \tag{128}$$

The *Parameterized Quantum of Time* is defined as follows:

$$\Delta t_{min,g}^*(\chi) = \Delta t_{min,\chi}^* = \frac{\Delta r_{min,\chi}^*}{c} \qquad (129)$$

Taking into account *(77)*, *(127)* and *(128)*, from the previous we obtain:

$$\frac{\Delta t_{min,\chi}^*}{\Delta t_{min,\pi/2}^*} = \frac{\Delta r_{min,\chi}^*}{\Delta R_{min}^*} = \frac{\Delta t_{min,\chi}}{\Delta t_{min}} = \frac{\Delta t_{min,\chi}}{\Delta t_{min,\pi/2}} = \sin\chi \qquad (130)$$

$$= \frac{1}{\eta_{\Delta r_{min}}^*} = \sqrt{1 - \frac{K}{R_K^*}}$$

It is worth highlighting how, from *(73)*, *(87)* and *(128)*, denoting with γ the so-called *Relativistic Factor*, we have:

$$\eta_{\Delta r_{min}}^* = \eta_{\Delta r_{min}} = \frac{1}{\sin\chi} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma \qquad (131)$$

V. METRICS

1. Initial "Flat" Metric (no singularity)

We can immediately write the following general metric:

$$ds^{*2} = c^2 dt^{*2} - dR_p^{*2} - R_c^{*2}(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (132)$$

Bearing in mind the definition of Predicted Radius provided by *(114)*, we have:

$$\lim_{\chi\to\pi/2} \frac{X_K^*}{R_K^*} = \lim_{\chi\to\pi/2} \frac{r_K^*}{R_K^*} = \lim_{\chi\to\pi/2} \sin\chi = 1 \qquad (133)$$

Consequently, far from the origin, Predicted Radius and Radial Coordinate are interchangeable. We can write:

$$R_c^* \cong R_K^* \qquad (134)$$

Now, we evidently have:

$$\lim_{\chi\to\pi/2} \sqrt{1 + \frac{1}{4\tan^2\chi}} = 1 \qquad (135)$$

Far from the origin, therefore, by virtue of *(119)*, Proper Radius and Radial Coordinate are interchangeable:

$$dR_p^* \cong dR_K^* \qquad (136)$$

Finally, far from the origin, *(132)* becomes:

$$ds^{*2} = c^2 dt^{*2} - dR_K^{*2} - R_K^{*2}(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (137)$$

It is fundamental to underline how the approximation in *(134)* prevents the Predicted Radius from assuming a null value. In detail, by virtue of *(109)*, we have:

$$R_{c,min}^*(\chi) = R_c^*(0) = R_K^*(0) = K \qquad (138)$$

2. Schwarzschild-Like Metric: Conventional Derivation

As is well known, the General Static, Spherically (and Time) Symmetric Solution can now be written as follows:

$$ds^2 = A(R_K^*)c^2 dt^2 - B(R_K^*)dR_K^{*2}$$
$$- C(R_K^*)(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (139)$$
$$A(R_K^*), B(R_K^*), C(R_K^*) > 0$$

Let us carry out the following position [11]:

$$\sqrt{C(R_K^*)} = R_K^* \qquad (140)$$

Thanks to the previous, *(139)* can be written as follows:

$$ds^2 = A^*(R_K^*)c^2 dt^2 - B^*(R_K^*)dr^2$$
$$- R_K^{*2}(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (141)$$
$$A^*(R_K^*), B^*(R_K^*) > 0$$

As for the *Metric Tensor*, from *(141)* we obtain:

$$g_{ij} = \begin{bmatrix} A^*(R_K^*) & 0 & 0 & 0 \\ 0 & -B^*(R_K^*) & 0 & 0 \\ 0 & 0 & -R_K^{*2} & 0 \\ 0 & 0 & 0 & -R_K^{*2}\sin^2\theta \end{bmatrix} \qquad (142)$$

$$g^{ij} = \begin{bmatrix} \dfrac{1}{A^*(R_K^*)} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{B^*(R_K^*)} & 0 & 0 \\ 0 & 0 & -\dfrac{1}{R_K^{*2}} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{R_K^{*2}\sin^2\theta} \end{bmatrix} \qquad (143)$$

Let's deduce the *Christoffel Symbols*. Generally, we have:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right) \qquad (144)$$

The indexes run from *0* to *3*. Clearly, *0* stands for *t*, *1* for *r*, *2* for *θ*, and *3* for *φ*.

Setting *k=0*, from *(142)*, *(143)* and *(144)*, we obtain:

$$\Gamma_{01}^0 = \Gamma_{10}^0 = \frac{1}{2A^*}\frac{dA^*}{dR_K^*} \qquad (145)$$

All the other symbols (if *k=0*) vanish.

Setting *k=1*, from *(142)*, *(143)* and *(144)*, we obtain:

$$\Gamma_{00}^1 = \frac{1}{2B^*}\frac{dA^*}{dR_K^*}$$
$$\Gamma_{11}^1 = \frac{1}{2B^*}\frac{dB^*}{dR_K^*}, \; \Gamma_{12}^1 = -\frac{R_K^*}{B^*}, \; \Gamma_{13}^1 = -\frac{R_K^*}{B^*}\sin^2\theta \qquad (146)$$

All the other symbols (if *k=1*) vanish.

Setting *k=2*, from *(142)*, *(143)* and *(144)*, we obtain:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{R_K^*}, \; \Gamma_{33}^2 = -\sin\theta\cos\theta \qquad (147)$$

All the other symbols (if *k=2*) vanish.

Setting *k=3*, from *(142)*, *(143)* and *(144)*, we obtain:

$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{R_K^*}, \; \Gamma_{23}^3 = \Gamma_{23}^3 = \frac{1}{\tan\theta} \qquad (148)$$

All the other symbols (if *k=3*) vanish.

Let's now deduce the components of the *Ricci Tensor*. Generally, with obvious meaning of the notation, we have:

$$R_{ij} = \frac{\partial \Gamma_{ik}^k}{\partial x^j} - \frac{\partial \Gamma_{ij}^k}{\partial x^k} + \Gamma_{ik}^l \Gamma_{jl}^k - \Gamma_{ij}^l \Gamma_{kl}^k \qquad (149)$$

By means of some simple mathematical passages, omitted for brevity, we obtain all the non-vanishing components:

$$R_{00} = -\frac{1}{2B^*}\frac{d^2A^*}{dR_K^{*2}} + \frac{1}{4B^*}\frac{dA^*}{dR_K^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} + \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right)$$
$$- \frac{1}{R_K^* B^*}\frac{dA^*}{dR_K^*} \qquad (150)$$

$$R_{11} = \frac{1}{2A^*}\frac{d^2 A^*}{dR_K^{*2}} - \frac{1}{4A^*}\frac{dA^*}{dR_K^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} + \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right)$$
$$- \frac{1}{R_K^* B^*}\frac{dB^*}{dR_K^*} \tag{151}$$

$$R_{22} = \frac{1}{B^*} + \frac{R_K^*}{2B^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} - \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right) - 1 \tag{152}$$

$$R_{33} = \sin^2\theta\left[\frac{1}{B^*} + \frac{R_K^*}{2B^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} - \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right) - 1\right]$$
$$= \sin^2\theta\, R_{22} \tag{153}$$

If we denote with *R* the *Ricci Scalar* and with T_{ij} the generic component of the *Stress-Energy Tensor*, the *Einstein Field Equations* [9] [12] can be written as follows:

$$R_{ij} - \frac{1}{2}R g_{ij} = \frac{8\pi G}{c^4}T_{ij} \tag{154}$$

If we impose that, outside the mass that produces the field, there is the "absolute nothing" (neither matter nor energy), the first member of *(154)*, that represents the so-called *Einstein Tensor*, must vanish. Consequently, we have:

$$R_{ij} - \frac{1}{2}R g_{ij} = 0 \tag{155}$$

From *(155)*, exploiting the fact that the Einstein Tensor and the Ricci Tensor are trace-reverse of each other, we have:

$$R_{ij} = 0 \tag{156}$$

From *(150)*, *(151)* and *(156)*, we immediately obtain:

$$-\frac{1}{2A^* B^*}\frac{d^2 A^*}{dR_K^{*2}} + \frac{1}{4A^* B^*}\frac{dA^*}{dR_K^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} + \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right)$$
$$- \frac{1}{R_K^* A^* B^*}\frac{dA^*}{dR_K^*} = 0 \tag{157}$$

$$\frac{1}{2A^* B^*}\frac{d^2 A^*}{dR_K^{*2}} - \frac{1}{4A^* B^*}\frac{dA^*}{dR_K^*}\left(\frac{1}{A^*}\frac{dA^*}{dR_K^*} + \frac{1}{B^*}\frac{dB^*}{dR_K^*}\right)$$
$$- \frac{1}{R_K^* B^{*2}}\frac{dB^*}{dR_K^*} = 0 \tag{158}$$

From *(157)* and *(158)*, we have:

$$\frac{dB^*}{B^*} = -\frac{dA^*}{A^*} \tag{159}$$

$$B^* = \frac{K_1}{A^*} \tag{160}$$

The value of the constant K_1 can be deduced by imposing that, at infinity, the Flat Metric in *(137)* must be recovered. In other terms, we must impose the following condition:

$$\lim_{R_K^* \to \infty} A^*(R_K^*) = \lim_{r \to \infty} B^*(R_K^*) = 1 \tag{161}$$

From *(160)*, taking into account *(161)*, we obtain:

$$B^* = \frac{1}{A^*} \tag{162}$$

$$g_{00}g_{11} = -1 \tag{163}$$

From *(152)* and *(156)* we have:

$$A^* + \frac{R_K^* A^*}{2}\left[\frac{1}{A^*}\frac{dA^*}{dR_K^*} - A^*\frac{d}{dR_K^*}\left(\frac{1}{A^*}\right)\right] - 1 = 0 \tag{164}$$

$$A^* + R_K^*\frac{dA^*}{dR_K^*} - 1 = \frac{d}{d}R_K^*(rA^*) - 1 = 0 \tag{165}$$

$$A^* = 1 + \frac{K_2}{R_K^*} \tag{166}$$

Now, if ϕ represents the *Gravitational Potential*, for an arbitrary metric we have:

$$g_{00} = \left(1 + \frac{\phi}{c^2}\right)^2 \tag{167}$$

The value of K_2 can be directly deduced by resorting to the so-called *Weak Field Approximation*:

$$\left(1 + \frac{\phi}{c^2}\right)^2 \cong 1 + 2\frac{\phi}{c^2} \tag{168}$$

Far from the source from *(97)*, *(110)* and *(168)* we have:

$$A^* = g_{00} \cong 1 + 2\frac{\phi}{c^2} = 1 - \cos^2\chi = 1 - \frac{K}{R_K^*} \tag{169}$$

If we set $K=R_s$, the foregoing can be written as follows:

$$A^* = 1 - \frac{2GM_{tot}}{c^2 R^*} = 1 - \frac{R_s}{R^*} \tag{170}$$

From *(162)* and *(169)*, we have:

$$B^* = \frac{1}{1 - \frac{K}{R_K^*}} \tag{171}$$

If we set $K=R_s$, the previous can be written as follows:

$$B^* = \frac{1}{1 - \frac{R_s}{R_K^*}} \tag{172}$$

At this point, the Schwarzschild-Like Metric can be immediately written by substituting into *(141)* the values of A^* and B^* deduced, respectively, in *(169)* and *(171)*.

3. Schwarzschild-Like Metric: Alternative Derivation
According to our model, taking into account *(106)* and *(115)*, from *(137)* we can deduce, in case of Singularity, the following solution:

$$ds_g^{*2} = c^2 dt^{*2} - dR_K^{*2} - R_K^{*2}\sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{173}$$

The previous represents an analytic solution, built without taking into account the modified value of the Space-Quantum. The above-mentioned condition is expressed by means of g_{00}, the value of which is manifestly unitary: Space and Time Quanta, in fact, are related to each other by means of *(129)*.

Obviously, t^* represents the proper time (the time measured by an observer ideally placed at infinity, where the singularity has no longer effect).
We can rewrite *(173)* in the underlying form:

$$ds_g^{*2}\big|_{g_{00}=1} = c^2 dt^{*2} - dR_{p,g}^{*2}\big|_{g_{00}=1}$$
$$- R_{c,g}^{*2}\big|_{g_{00}=1}(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{174}$$

In other terms, we have carried out the following positions:

$$R_{c,g}^*\big|_{g_{00}=1} = R_c^*\sin\chi = R_K^*\sin\chi \tag{175}$$

$$dR_{p,g}^*\big|_{g_{00}=1} = dR_p^* = dR_K^* \tag{176}$$

Now, from *(130)* we immediately obtain:

$$\Delta t_{min,g}^*(\chi) = \Delta t_{min,\chi}^* = \frac{\Delta t_{min,\pi/2}^*}{\eta_{\Delta r_{min}}^*} = \frac{\Delta t_{min}^*}{\eta_{\Delta r_{min}}^*} \tag{177}$$

$$= \Delta t_{min}^* \sqrt{1 - \frac{K}{R_K^*}}$$

In the light of the previous, we can write:

$$dt_g^* = \frac{dt_{\pi/2}^*}{\eta_{\Delta r_{min}}^*} = \frac{dt^*}{\eta_{\Delta r_{min}}^*} = dt^* \sqrt{1 - \frac{K}{R_K^*}} \tag{178}$$

From *(175)*, taking into account *(120)* and *(127)*, we have:

$$R_{c,g}^*\big|_{g_{00}=1} = \mathcal{N} \sin\chi\, \Delta r_{min,\pi/2} = \mathcal{N}\Delta r_{min,\chi} \tag{179}$$

Exploiting *(134)* and *(175)*, we can temporarily introduce to following *Non-Dimensional (Normalized) Coordinates*:

$$\bar{R}_c^* = \frac{R_c^*}{\Delta R_{min}^*} = \frac{R_c^*}{\Delta r_{min,\pi/2}^*} = \mathcal{N} \tag{180}$$

$$\bar{R}_{c,g}^*\big|_{g_{00}=1} = \frac{R_{c,g}^*}{\Delta r_{min,\chi}^*} = \mathcal{N} \tag{181}$$

Evidently, the value of the Predicted Radius, as long as it is expressed in terms of Space-Quanta, can be regarded as being constant. Consequently, from *(180)* and *(181)* we can banally write:

$$\bar{R}_c^* = \bar{R}_{c,g}^*\big|_{g_{00}=1} \tag{182}$$

Now, if we replace *dt** with *dt_g**, taking into account *(178)*, we obtain a new value for *g₀₀*:

$$g_{00} = 1 - \frac{K}{R_K^*} = \frac{1}{\eta_{\Delta r_{min}}^{*2}} \tag{183}$$

The value of g_{00} reveals how we measure time (which is still considered as being absolute) and space and nothing else. In other words, we have simply changed the *Units of Measurement* (we have modified the Scale Factor).

By virtue of *(183)*, we can rewrite *(173)* by changing the Scale Parameter:

$$ds_g^{*2}\big|_{g_{00}=1/\eta^2} = \left(1 - \frac{K}{R_K^*}\right)c^2 dt^{*2} - dR_{p,g}^{*2}\big|_{g_{00}=1/\eta^2} \tag{184}$$

$$- R_{c,g}^{*2}\big|_{g_{00}=1/\eta^2}(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

From *(175)*, *(176)* and *(183)*, we can write, with obvious meaning of the notation, the following:

$$R_{c,g}^*\big|_{g_{00}=1/\eta^2} = \eta_{\Delta r_{min}}^* R_{c,g}^*\big|_{g_{00}=1} = R_K^* = R_{c,g}^* \tag{185}$$

$$dR_p^*\big|_{g_{00}=1/\eta^2} = \eta_{\Delta r_{min}}^* dR_{p,g}^*\big|_{g_{00}=1} = \eta_{\Delta r_{min}} dR_K^*$$

$$= \frac{dR_K^*}{\sqrt{1 - \frac{K}{R_K^*}}} = dR_p^* \tag{186}$$

We can finally write the so-called *Droste Solution* [13]:

$$ds_g^{*2}\big|_{g_{00}=\frac{1}{\eta^2}} = \left(1 - \frac{K}{R_K^*}\right)c^2 dt^{*2} - \frac{dR_K^{*2}}{1 - \frac{K}{R_K^*}} \tag{187}$$

$$- R_K^{*2}(d\theta^2 + \sin^2\theta\, d\varphi^2) = ds_g^{*2}$$

$$R_K^* > K$$

The Singularity is not a point, but a 2-Sphere characterized by a radius equal to *K*. However, this strange phenomenon is anything but real, since it is clearly and exclusively caused by the approximation in *(134)*. According to the new scenario, the value of the *Escape Speed* is now provided by *(104)*: it is easy to verify how this value formally coincides with the one that can be derived by resorting to the *Geodesic Equation*.

As for the *New Proper Radius*, we have:

$$R_{p,g}^* = \int \frac{dR_K^*}{\sqrt{1 - \frac{K}{R_K^*}}} = \int \frac{\sqrt{(R_K^* - K) + K}}{\sqrt{R_K^* - K}} d(R_K^* - K) \tag{188}$$

$$= 2\int \sqrt{\left(\sqrt{R_K^* - K}\right)^2 + \left(\sqrt{K}\right)^2}\, d\left(\sqrt{R_K^* - K}\right)$$

We have just found an integral of the following kind:

$$\int \sqrt{y^2 + \left(\sqrt{K}\right)^2}\, dy = \frac{K}{2}\ln\left(y + \sqrt{y^2 + K}\right) \tag{189}$$

$$+ \frac{y}{2}\sqrt{y'^2 + K} + C_K$$

Consequently, from *(188)* and *(189)* we have:

$$R_{p,g}^* = \int \frac{dR_K^*}{\sqrt{1 - \frac{K}{R_K^*}}} = \ln\left(\sqrt{R_K^*} + \sqrt{R_K^* - K}\right) \tag{190}$$

$$+ \sqrt{R_K^*(R_K^* - K)} + C_K$$

As for the constant *C_k* we have:

$$R_p^*(K) = 0 \Rightarrow C_K = -K \ln\sqrt{K} \tag{191}$$

Finally, from *(190)* and *(191)* we have:

$$R_{p,g} = K \ln\left(\frac{\sqrt{R_K^* - K} + \sqrt{R_K^*}}{\sqrt{K}}\right) + \sqrt{R_K^*(R_K^* - K)} \tag{192}$$

The previous, by virtue of *(117)*, can be written as follows:

$$R_{p,g} = K \ln\left(\frac{\sqrt{x_K^*} + \sqrt{x_K^* + K}}{\sqrt{K}}\right) + \sqrt{x_K^*(x_K^* + K)} \tag{193}$$

4. Generalization

Taking into account *(117)*, we have:

$$\lim_{\chi \to \pi/2} \frac{x_K^*}{R_K^*} = \lim_{\chi \to \pi/2} \frac{R_K^* - K}{R_K^*} = 1 \tag{194}$$

By virtue of the previous, we can write:

$$\lim_{\chi \to \pi/2} \frac{x_K^{*a}}{R_K^{*a}} = \lim_{\chi \to \pi/2} \frac{x_K^{*a} + K^a}{R_K^{*a}} \tag{195}$$

$$= \lim_{\chi \to \pi/2} \frac{(R_K^* - K)^a + K^a}{R_K^{*a}} = 1$$

$$a \in \mathcal{R}^+$$

Therefore, far from the source, we obtain:

$$R_K^{*a} \cong (R_K^* - K)^a + K^a \tag{196}$$

$$R_K^* \cong \sqrt[a]{(R_K^* - K)^a + K^a} = \sqrt[a]{x_K^{*a} + K^a} = R_{K,a}^* \tag{197}$$

Evidently, moreover, we have:

$$\chi = 0 \Rightarrow R_{K,a}^* = K \tag{198}$$

$$\frac{dR_{K,a}^*}{d\chi} = \left(x_K^{*a} + K^a\right)^{\frac{1-a}{a}} x_K^{*a-1} \frac{dx_K^*}{d\chi} > 0 \tag{199}$$

$$0 < \chi < \frac{\pi}{2}$$

From *(195)*, *(198)* and *(199)* we deduce how the *New Parametric Coordinate* defined in *(197)* and the one defined in *(109)* are fully interchangeable (since they behave exactly the same way). In other terms, we have:

$$R_{K,a}^* \cong R_K^* \tag{200}$$

Taking into account the foregoing, by setting *a=1* in *(197)*, from *(187)* we obtain:

$$ds_g^{*2} = \left(1 - \frac{K}{x_K^* + K}\right)c^2 dt^{*2} - \frac{dx_K^{*2}}{1 - \frac{K}{x_K^* + K}}$$
$$- (x_K^* + K)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{201}$$
$$x_K^* > 0$$

$$ds_g^{*2} = \frac{c^2 dt^{*2}}{1 + \frac{K}{x_K^*}} - \left(1 + \frac{K}{x_K^*}\right)dx_K^{*2}$$
$$- (x_K^* + K)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{202}$$
$$x_K^* > 0$$

The previous represents the original form of the so-called *Brillouin Solution* [14].
From *(187)*, by setting *a=3* in *(197)*, we have:

$$R_{K,3}^* = \sqrt[3]{x_K^{*3} + K^3} \tag{203}$$

By substituting the previous into *(187)*, we can finally obtain the real *Schwarzschild Form* [15].

VI. GRAVITATIONAL REDSHIFT

If we impose the Conservation of Energy, we can write, with obvious meaning of the notation, the following:

$$E_{photon,\chi} = h_\chi \nu_\chi = h_{\pi/2}\nu_{\pi/2} = E_{photon,\pi/2} \tag{204}$$

From the previous, by virtue of *(75)*, we obtain:

$$\frac{h_\chi}{h_{\pi/2}} = \frac{\nu_{\pi/2}}{\nu_\chi} = \sin^2\chi \tag{205}$$

If we impose the *Speed of Light Constancy*, we have:

$$c = \lambda_\chi \nu_\chi = \lambda_{\pi/2}\nu_{\pi/2} \tag{206}$$

The two foregoing relations allows to immediately define a *New Scale Parameter*:

$$\frac{\lambda_{\pi/2}}{\lambda_\chi} = \frac{h_{\pi/2}}{h_\chi} = \frac{1}{\sin^2\chi} = \eta_\lambda \tag{207}$$

According to the definition of *Gravitational Redshift* [9], usually denoted by *z*, from the previous we have:

$$z = \frac{\lambda_{\pi/2} - \lambda_\chi}{\lambda_\chi} = \frac{\lambda_{\pi/2}}{\lambda_\chi} - 1 = \eta_\lambda - 1 \tag{208}$$

From (131) and (207) we have:

$$\lim_{\chi \to \pi/2} \frac{\eta_{\Delta r_{min}}}{\eta_\lambda} = \lim_{\chi \to \pi/2} \frac{\eta_{\Delta r_{min}}^*}{\eta_\lambda} = \lim_{\chi \to \pi/2} \sin\chi \tag{209}$$
$$= \lim_{v \to 0} \sqrt{1 - \left(\frac{v}{c}\right)^2} = 1$$

Consequently, far from the source, we can write:

$$z \cong \eta_{\Delta r_{min}} - 1 = \frac{1}{\sin\chi} - 1 \tag{210}$$

From the foregoing, taking into account *(110)*, we have:

$$z = \frac{\lambda_\infty - \lambda_{R^*}}{\lambda_{R^*}} = \frac{1}{\sqrt{1 - \frac{K}{R_K^*}}} - 1 \tag{211}$$

If we set *K=Rₛ*, according to *(2)* and *(109)*, the previous can be written in the following well-known form:

$$z = \frac{1}{\sqrt{1 - \frac{2GM_{tot}}{c^2 R^*}}} - 1 \tag{212}$$

VII. BRIEF CONCLUSIONS

The coordinate deduced in *(109)*, which appears both in the metrics and at the denominator of the pseudo-Newtonian relation we have obtained for the gravitational potential, does not represent a real distance nor a real radius of curvature. In fact, it is clear how the expression of the above-mentioned coordinate arises from a banal parameterization, by means of which we are able to write the initial "Flat" Metric in *(137)*. From the latter, it is possible to derive an infinite set of Schwarzschild-like Metrics, suitable for non-rotating and non-charged "Black Holes", without resorting to Relativity. According to the simple model herein proposed, the minimum value for the coordinate in *(109)* equates the Schwarzschild Radius. When this coordinate equates the Schwarzschild radius, both the Proper Radius and the Forecast Radius are equal to zero: in other terms, we are exactly placed in correspondence of the "Singularity".

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